FAST AND ACCURATE PROPAGATION OF COHERENT LIGHT

RYAN D. LEWIS, GREGORY BEYLKIN, AND LUCAS MONZÓN

Abstract. We describe a fast algorithm to propagate, for any user-specified accuracy, a time-harmonic electromagnetic field between two parallel planes separated by a linear, isotropic, and homogeneous medium. The analytic formulation of this problem (circa 1897) requires the evaluation of the so-called Rayleigh-Sommerfeld integral. If the distance between the planes is small, this integral can be accurately evaluated in the Fourier domain; if the distance is very large, it can be accurately approximated by asymptotic methods. In the large intermediate region of practical interest

nu r a) asured, or produced by a computational procedure such as phase recovery such analytic expansions and yield only a limit accuracy. urth r o nt on the stopic in $\S C$ of the online supplement.

h n or an a urat propa at on a) or th arss n ar as su h as o pu tational holography component design and antinna design A . part $u \text{ar}$, interesting application area is X-ray diffraction more comparation area is $x \text{ar}$, and $y \text{ar}$ related techniques where on attempts to order and a comparate of α microscopic sample. p) row asuremts other a nitude of the magnitude of its magnitude of its difference in the magnitude of its difference problems are usually solved by the ratio vector of the sthat in μ a light propagation a) or the herefore, the accuracy of the propagation algorithm ultimately limits the accuracy of the reconstructed in a speed of a propagation algorithm is obvous) also of critical importance of applications employees in the state \mathbf{a} is the state method in the state methods. h numerical algorithms that we use are designed to yield and user-specified \ln mumer-specified and \ln

a ura his n u includes controlled a controlled accuracy in the rapid computation of integrals. h the sthat we power this purpose specifically the USFF and generally the USFFT and generally the USFFT and general h a) Gaussian qualratures or band-limited functions for an significantly improve functions (and significantly improve functions) can significantly improve functions of α and α is denoted functions. the pror and and accuracy of \mathbf{r} and \mathbf{r} and \mathbf{r} and \mathbf{r} is the standard methods for \mathbf{r} between \mathbf{r} and \mathbf{r} s §§A and B o the online supplement on \mathbb{R} .

 h pap r s or an as o , b ws h n ssar at h at a , p r , h nar s arrvw $n \S$ srbournwalgorth $n \S$ then suss its region of va) t $n \S$ In \S we provide severation in all x a ples then summarize our rsults n § B ntrounths nwalorth whope to stult a ura provemts in computational optical systems by essentially eliminating numerical errors.

2. Preliminaries

The Rayleigh-Sommerfeld Formula. h b hav or o a t \int harmonic $\begin{bmatrix} \text{or} & \text{or$ Gvn the boundary data $\boldsymbol{u} \cdot \boldsymbol{x}_t$ of \boldsymbol{x}_t we rewrite \boldsymbol{u}_t as

$$
u\cdot x,z \qquad \quad f\cdot y \quad K_z \quad \, x-y \quad \, dy,
$$

whrth $a \rightarrow \nightharpoonup$ is $c \rightarrow c$ in K_z (r) is vnb

$$
K_{z} r
$$
 $\frac{e^{i2 z} \overline{1+(r/z)^2}}{iz} \frac{1}{\sqrt{r/z^2}} \frac{1}{z^2-r/z^{\frac{3}{2}}}, r \ge 0$

D not n the Four r transform of the boundary data as

$$
\mathbf{f} \cdot \mathbf{p} \qquad \mathbf{f} \cdot \mathbf{x} \cdot e^{-i2 \cdot \mathbf{x} \cdot \mathbf{p}} \, \mathbf{dx},
$$

w wrt. nth Four r o an as

$$
\mathbf{u} \cdot \mathbf{x}, \mathbf{z} \qquad \mathbf{f} \cdot \mathbf{p} \quad \mathbf{K}_{\mathbf{z}} \qquad \mathbf{p} \quad e^{\mathbf{i} 2 - \mathbf{x} \cdot \mathbf{p}} \mathbf{d} \mathbf{p},
$$

wh r the Fourier transform of the Rayleigh-Sommerfeld kernel (cf. et al. (cf. et al. et n sthrns vnb

$$
K_z
$$
 $e^{i2 z} \overline{1-z}$ \ge .

ur oa) s to valuate (2.2) accurately in such a way that the computational cost o s not n r as with the stance z It is contributed the spatial kernel K_z r s a h h) os jator un ton o r wh n z s s a) and that the Fourier domain kernel K_z sahighly oscillator untono when z star For an ph s all interesting the standard choices of the intermediate region, K_z (r) and K_z ar both h h ighly oscillator and the r thur a) of putation of $\mathbf u$ using ther or (2.2) or (2.4) improvided in \int_a^b we will show how to approximate with ontrolled error and then \sinh and a factor and algorithm to apply then rsulting approximate Green's summature of the mainly function to boundary data. Our algorithm mainly material $\lim_{n\to\infty}$ a r ss s the propagation problem for r intermediate and large values of z. For small values of z, it is well known that the problem may be solved using Fourier the set of \mathbf{z} is \mathbf{w} we show that the problem methods is solved using Fourier methods is \mathbf{z} an or vr and values of z, the problem may be solved using asymptotic methods s §§A and B o the online supplement on \sinh

Remark \overrightarrow{G} \overrightarrow{G} \overrightarrow{v} n the normal derivative of the boundary data

$$
\begin{array}{c c c c}\n\text{-}\mathbf{u} & \mathbf{x},\mathbf{z} & \mathbf{g} & \mathbf{x} \end{array}
$$

Lore $a \rightarrow bs$ or $u \cdot a$ or the $u \cdot a$ reads reads reads the Neumann problem reads reads not $u \cdot a$ or $t \cdot a$

$$
u \cdot x, z = -\int_{\mathbb{R}^2} g \cdot y \frac{e^{i2} R}{R} dy, \quad R = z^2 \quad x - y^{2-\frac{1}{2}}, \quad z > 0.
$$

th nor o at ons our approach salso applied is a in to valuat n

Slepian Functions. All ph s all realistic fields we wentually decay in spa an at the same time, are essentially band-limited in the Fourier domain. An appropriate mathematical description of such the such fields fields and his such fields $\mathbb R$ collaborators n \Box by onsignma space-limiting and band-limiting and band-li nt rajoprator and using its eigenfunctions to identify a class of μ in the functions that have ontrolled concentration in both the space and the Fourier domains. \mathfrak{D} and \mathfrak{D} t a) showed that this integral operator commutes with the differential operator o (ass a) ath at a)ph s s s r b n th pro at sph ro a) wave untons both operators share the same eigenfunctions.

For our purposes, we use eigenfunctions with $\text{onto } \mathcal{W}$ on $\text{intract on } n$ a square n the spatial domain and band-limited to a second-limited to a disk in the Fourier domain. The results in the Fourier domain. The spatial contract is spatially spatially spatially spatially spatially spatially spatially sp

 $\sum_{n=1}^{\infty}$ The Unequally Spaced Fast Fourier Transform. We need to evaluate transform. $\operatorname{tr-} \operatorname{su-} \operatorname{so-} \operatorname{th-} \operatorname{or}$

$$
\begin{aligned} \mathsf{M} \\ \mathsf{m} \ \mathsf{m}^{\prime} \ \mathbf{f} \ \ \mathbf{y}_{\mathsf{m} \mathsf{m}^{\prime}} \ \ \mathbf{e}^{i \mathbf{x} \cdot \mathbf{y}_{\mathsf{m} \mathsf{m}^{\prime}}} \\ \mathsf{m}, \mathsf{m}^{\prime} {=}1 \end{aligned}
$$

at output points $\mathbf{x}_{nn'}$ $\mathbf{x}_{n'}$, $\mathbf{x}_{n'}$ where n, n' , ..., N. The u is and beevaluated rapidly, for any user-specified accuracy , using the USFFT (see [11, 3, with o putational complexity $O \mathbb{N}^2$

phas that n the sr a uracy K is scaled by the propagation stan z s n th a n tu o th m \ a s \ z^{-1} a \ bn the optial axis. Inspired by the Fresnel Approximation, we rewrite the kernel as

$$
K_z\cdot r=-\frac{e^{i2-z}e^{i\frac{\pi}{z}r^2}}{iz}A_z\cdot r\ ,
$$

 $\leq r$

$$
A_{z} r \longrightarrow r/z^{2} \frac{i}{z^{2} \sqrt{r/z^{2}}} e^{i2 z (\frac{1}{1+(r/z)^{2}}-1-\frac{1}{2}(r/z)^{2})}.
$$

wh r c s the bandlich to the input unit on f sn the bandlich \mathfrak{c}' we a rt the integrals in the integral or a service accuracy quantity quantures from α h or

L t $y_{mm'}$ y_m, $y_{m'}$ A m, m', ..., M b th $M \times M$ t nsor prout r o qua ratur no swith the corresponding quadrature weights $m \cdot m'$. We sell the set of m an $N \times N$ grid output locations $x_{nn'}$ $x_{n'}$, $x_{n'}$ w n, n' , ..., N then applie the quantum rotation of the theory is to the network of the matter and w and w is $n \times N$ then apply the quadrature from h or approximation to the output of λ at the desired locations as

$$
u_{nn'} = \frac{e^{i2 z}}{iz} = \frac{1}{2} w_{mn'm'} m_{nn'm'} m_{nn'm''} f_{nm'm''} e^{i2 \frac{z x_{nn'} \cdot y_{mm'}}{iz}}.
$$

In the $\textbf{N}\times\textbf{N}\times\textbf{M}\times\textbf{M}$ our
th-order tensors $\textbf{T}^{(\)}$

Lemma 4. Let $\begin{pmatrix} 1 \ 0 \end{pmatrix}$, **U**_{nq}, and $V_{mq}^{(1)}$, where $\begin{pmatrix} 1 & 1 \ 0 & 1 \end{pmatrix}$, ..., L,

Theorem 5. The error of computing the field u from $\overline{}$ using $\overline{}$ is bounded by

$$
|u_{1}\mathbf{x}_{nn^{\prime}}_{l}z-u_{nn^{\prime}}|\quad\frac{K\quad\text{o}\quad R\quad f_{1}}{z}.
$$

In the expression or $u_{nn'}$ in (3.15) allows us to valuate the field rapidly. We first apply $\mathbf{Q}^{(-)}_{\mathsf{mm}'\mathsf{r}}$ as a pr $]$ a tor to the input samples f (ymm'

o illustrat the difference between \mathbf{w}_{max} and \mathbf{z}_{min} or our the and \mathbf{w}'_{max} and z'_{min} or the Fr snel approximation, let us hoos z^{-3} I a wavelengths then at r propagating $z \times 6$

Figure 5.5. Co parson oth a ntu oth ϕ or a oal point \circ o the optical axis of puteent by our algorithm correct to $ts \rightarrow t$ and by the Fresnel approximation (right) of enhance contrast we plot the square root of the magnetic m $|{\bf u} \times_1, {\bf x}_2 |^{1/2}$ h Fr sn) approximation shifts the location of the focal spot and blurs the boundaries between the manufold and side of \mathcal{I} a so the bottom-right plot in Figure

n Fur to bttra) nth pasoth solid and ashed lines. In ortunately, our xample shows that the Fr snu expresses the short result in \mathbb{R}^n shape incorrectly computed shape in \mathbb{R}^n o the o alspot, nadd to its post on o pare the nulls between the main $\label{eq:1} \begin{array}{lll} \text{in} & \text{in}$

Representative Examples of Computational Cost. In o putational

Conclusions

have srb a ast alorth or the propagation of coherent light between para led planes separated by a linear sotropic and homogeneous medium. In contrast to urr nt a) or the sour a) or the a heves and user-specified accuracy. As a ons qun w an rap i and accurately compute the in non-paraxial regions, i.e., regions are regions for the optical axis, with computational complexity proportional to that of the FFT. The overall result is a fast algorithm that can achieve an user-specified accuracy over a large computational domain.

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