APPM GRADUATE PRELIMINARY EXAMINATION
PARTIAL DIFFERENTIAL EQUATIONS – SOLUTIONS

Thursday August 24, 2017, 10AM –1PM

There are five problems. Solve any four of the five problems. Each problem is worth 25 points.

On the front of your bluebook please write: (1) your name and (2) a grading table. Please start each problem with a new page. Text books, notes, calculators are NOT permitted. A sheet of convenient formulae is provided.

1. (**First order equations**)

(a) (18 points)
Solve the first-order initial value problem

$$e^x \frac{\partial u}{\partial x} + ($$

(b) When = 0, the PDE reduces to the ODE

$$(t + 1)\frac{du}{dt} = \quad u,$$

its general solution is

$$u = u_0(x) \quad (t + 1)$$

with $u_0(x)$ independent of t $_0$(

n =

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(b) As $t \to \infty$, the solution $u(x,t) \to 0$. Since $\phi(t)$ is nonzero only on a finite interval of t, the approximate solution for large t can be written as

$$u(x,t) \approx \frac{e^{\int_0^\infty \phi(\tau)d\tau}}{x + x_0} \, b_1 e^{-\pi^2 t/L^2} \sin\frac{\pi x}{L},$$

i.e. it is determined by the lowest mode $k_1 = \pi/L$. The characteristic time of convergence to zero is $\sim L^2/(\pi^2)$ and the time T is determined by $u(L/2, t+T) = u(L/2, t)/2$, i.e.

$$T \approx \frac{L^2}{\pi^2}\ln 2.$$

3. (**Fourier series**)

(a) (10 pts)

Show that the pointwise convergent series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{1/2}}$$

cannot converge uniformly to a square integrable function f in $[-\pi, \pi)$.

(b) (15 pts)

Let $f(x)$ be 2π periodic and piecewise smooth. Prove that its Fourier series converges uniformly and absolutely to f.

Solution:

(a) Suppose the series converged uniformly to a square integrable function f. The Fourier coefficients of f are

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(f$$ 1327 23887 Td [ ]JF64 79701 Tf 6982 0 Td [ ]JF59 1

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f(x) in 2 (f nx 1327 23887 Td [ ]JF64 79701 Tf 6982 0 Td [ ]JF59 119552 Tf F53 119552 Tf1321 0 Td [494906

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4. (**Wave type equations**)

Consider

$$u_{tt} - c^2 u_{xx} + a u_t + \frac{a^2}{4} u = 0\,, \qquad 0 \le x \le L\,, \quad t > 0\,, \tag{4}$$
$$u(x,0) = f(x)\,, \quad u_t(x,0) = g(x)\,, \quad u(0,t) = u(L,t) = 0\,,$$

where $f(x), g(x)$ are integrable and $c > 0$ and $a > 0$ are constants.

(a) (15 points)

Solve the above initial boundary value problem.

Hint: Look for solutions of the form $u(x,t) = e^{-\frac{a}{2}t} w(x,t)$.

(b) (5 points)

Derive the energy relation

$$\frac{dE}{dt} = -2a \int_0^L u_t^2 dx\,, \tag{5}$$
$$E(t) = \int_0^L \left(u_t^2 + u_x^2 + \frac{a^2}{4} u^2 \right) dx\,.$$

What physical effect do the additional terms au_t and $a^2u/4$ in (4) represent?

(c) (5 points)

Using energy relation (5), prove that the solution found in part **(5 po14.968-1.793Td[(t)]TJ/3h7r**

) 0x0). -a

The boundary conditions $u(0,t) = u(L,t) = 0$ imply $u_t(0,t) = u_t(L,t) = 0$. Performing integration-by-parts on the second term and applying these boundary conditions yields the desired energy relation

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left(u_t^2 + u_x^2 + \frac{a^2}{4}u^2 \right) dx = -a\int_0^L u_t^2 dx.$$

The energy $E(t)$ is non-increasing in time, i.e. $E(t_2) \le E(t_1)$ for $t_2 > t_1$, indicating some dissipative force (e.g. friction, vibration) is modeled by the terms au_t and $a^2u/4$.

(c)

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