Dⓔpartmⓔnt of pppliⓔd Mathⓔmatics
rⓔliminary Examination in Numⓔrical nalysis
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Solutions:

1. **Root Finding.**

   (a) Let the root be $x =$: We subtract from both sides of $x_{n+1} = x_n$ $f(x_n)$)

3. **Interpolation/Approximation.**

(a) $p_n(x) = \sum_{k=0}^{n} L_k(x) f_k;$ here $L_k(x) = \frac{(x \; x_0) \; ::: \; (x \; x_{k \; 1})(x \; x_{k+1}) \; ::: \; (x \; x_n)}{(x_k \; x_0) ::: (x_k \; x_{k \; 1})(x_k \; x_{k+1}) ::: (x_k \; x_n)}.$

(b) Suppose there are t o different pol nomials $p_n(x)$ and $q_n(x)$ that both take the values f_k at node locations $x_k; \; k = 0; 1; ::: ; n:$ The difference $p_n(x) \quad q_n(x)$ is again a pol nomial of degree n but ith $n + 1$ zeros, sho ing that it must be identicall zero, in conflict ith the assumption that $p_n(x)$ and $q_n(x)$ ere different.

(c) Each of the follo ing three approaches ill sho that, for $n + 1$ nodes, the pol nomial degree ill be $2n + 1$.

(i) **Direct solution of linear system:** Let the Hermite pol nomial be $H_{2n+1}(x) = a_0 + a_1 x + a_2 x^2 + ::: + a_{2n+1} x^{2n+1}:$ Imposing all the $2n + 2$ requirements gives a square $(2n+2) \quad (2n+2)$ linear s stem of the follo ing structure for the coefficients:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & ::: \\ 1 & x_1 & x_1^2 & x_1^3 & ::: \\ ::: & & & & ::: \\ & & & & \\ 0 & 1 & 2x_0 & 3x_0^2 & ::: \\ 0 & 1 & 2x_1 & 3x_1^2 & ::: \\ ::: & & & & ::: \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \\ \\ \vdots \\ a_{2n+1} \end{bmatrix} = \begin{bmatrix} f_0 \\ \vdots \\ f_n \\ f_0' \\ \vdots \\ f_n' \end{bmatrix}:$$

(ii) **Based on Lagrange interpolation:** With $L_k(x)$ denoting the Legendre kernel, the pol nomials $\hbar_i(x)$frpTf-0.]TJ/F237.9701Tf3.2930Td[(x)]TJ/F22+ 1x

$x_k x$

4. **Linear algebra**

(a) Since A is an antis mmetric matrix, its eigenvalues are purel imaginar , or zero. Since it is a matrix ith real entries, the roots of the characteristic pol nomial come in pairs (if the are complex-valued). For odd-sized matrix these t o conditions force at least one of the eigenvalues to be zero.

(b) For even-sized matrix the product of a pair of complex-valued eigenvalues is al a s positive and the conclusion follo s.

(b) For an explicit multistep method, the equation for the roots of the characteristic pol -
nomial has the form

$$(u) = u^s + \text{lo\ \ er order terms} = 0.$$

Since the pol nomial can be ritten in terms of its roots as

$$(u) = (u \quad u_1)(u \quad u_2) \dots (u \quad u_s);$$

and in the region of absolute stabilit all roots $|u_k| \quad 1$, e conclude that, in that region, all coefficients of the pol nomial are bounded (independentl of h). Ho ever, if the region of absolute stabilit is unbounded, then some of the coefficients of ill become where

$$A = \frac{1}{2}(c\lambda)^2 - \frac{1}{2}c\lambda; \quad B = (c\lambda)^2 - 1 \text{ and } C = \frac{1}{2}(c\lambda)^2 + \frac{1}{2}c\lambda.$$

Using $\left(e^{ijkh_x}\right)_{j=0}^{N-1}$ as an eigenvector (with index $k = 0, \ldots, N-1$), we compute

$$\begin{aligned} Ae^{i(j+1)kh_x} - Be^{ijkh_x} + Ce^{i(j-1)kh_x} &= e^{ijkh_x}\left(Ae^{ikh_x} - B + Ce^{-ikh_x}\right) \\ &= e^{ijkh_x}\left[1 - (c\lambda)^2 + (c\lambda)^2\cos(kh_x) - ic\lambda\sin(kh_x)\right] \end{aligned}$$

Computing the absolute value of the eigenvalue $\lambda_k = 1 - (c\lambda)^2 + (c\lambda)^2\cos(kh_x) - ic\lambda\sin(kh_x)$, we have

$$\begin{aligned} |\lambda_k|^2 &= \left[1 - (c\lambda)^2 + (c\lambda)^2\cos^2(kh_x)\right]^2 + (c\lambda)^2\sin^2(kh_x) \\ &= \left[1 - (c\lambda)^2\sin^2(kh_x)\right]^2 + (c\lambda)^2\sin^2(kh_x). \end{aligned}$$

Setting $a = (c\lambda)^2$, $a > 0$ and $x = \sin^2(kh_x)$, $0 \leq x \leq 1$, as a function of x we have $(1 - ax)^2 + ax = 1 - ax + a^2x^2$. The condition $a \leq 1$ implies that

$$1 - ax + a^2x^2 \leq 1.$$

Thus, we obtain stability under the CFL condition $c\lambda \leq 1$ or $h_t/h_x \leq 1/c$.