1. (26 pts) Evaluate the integral.

   (a) $\displaystyle\int \frac{2x^2 - 3x + 10}{x^3 + 5x}\,dx$

   (b) $\displaystyle\int \frac{1}{(x^2 - 1)^{3/2}}\,dx$

Solution:

(a)

$$\int \frac{2x^2 - 3x + 10}{x^3 + 5x}\,dx = \int \left(\frac{A}{x} + \frac{Bx + C}{x^2 + 5}\right) dx$$

Solve $A\left(x^2 + 5\right) + x(Bx + C) = 2x^2 - 3x + 10$ to find the values $A = 2$, $B = 0$, and $C = -3$.

$$\int \frac{2x^2 - 3x + 10}{x^3 + 5x}\,dx = \int \left(\frac{2}{x} - \frac{3}{x^2 + 5}\right) dx$$

$$= \boxed{2\ln|x| - \frac{3}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C_1}$$

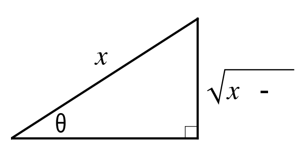
(b) Let $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$.

$$\int \frac{dx}{(x^2 - 1)^{3/2}} = \int \frac{\sec\theta\tan\theta}{(\sec^2\theta - 1)^{3/2}}\,d\theta = \int \frac{\sec\theta\tan\theta}{(\tan^2\theta)^{3/2}}\,d\theta$$

$$= \int \frac{\sec\theta\tan\theta}{\tan^3\theta}\,d\theta = \int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \csc\theta\cot\theta\,d\theta = -\csc\theta + C$$

$$= \boxed{-\frac{x}{\sqrt{x^2 - 1}} + C}$$



2. (20 pts) This problem has three parts.

Let $f(x) = 1 + \ln\left(\frac{x}{x+1}\right)$. Consider the integral $\int_1^4 f(x)\,dx$.

(a) Estimate the value of the integral using T_3, the trapezoidal approximation with $n = 3$ subintervals. Fully simplify your answer by combining logarithms.

(b) Given that $-\frac{3}{4} \le f''(x) \le -\frac{1}{50}$ for $1 \le x \le 4$, how large should n be to ensure that the approximation error for T_n is within 10^{-4}? Simplify your answer.

(c) Is the T_3 approximation found in part (a) an underestimate or overestimate? Justify your answer. (*Hint:* It is not necessary to find the exact value of the integral.)

Solution:

(a) Let $\Delta x = \frac{b-a}{n} = \frac{3}{3} = 1$. Then

$$\begin{aligned} T_3 &= \frac{1}{2}(\Delta x)\left(f(1) + 2f(2) + 2f(3) + f(4)\right) \\ &= \frac{1}{2}(1)\left[\left(1 + \ln\frac{1}{2}\right) + 2\left(1 + \ln\frac{2}{3}\right) + 2\left(1 + \ln\frac{3}{4}\right) + 1 + \ln\frac{4}{5}\right] \\ &= \frac{1}{2}\left[6 + \ln\left(\frac{1}{2}\cdot\frac{2^2}{3^2}\cdot\frac{3^2}{4^2}\cdot\frac{4}{5}\right)\right] = \frac{1}{2}\left[6 + \ln\frac{1}{10}\right] \\ &= 3 - \frac{1}{2}\ln 10 . \end{aligned}$$

(b) Let $K = \frac{3}{4}$, the maximum value of $|f''|$. Solve this inequality for n:

$$\begin{aligned} |E_T| \le \frac{K(b-a)^3}{12n^2} &< 10^{-4} \\ \frac{(3/4)(3^3)}{12n^2} &< \frac{1}{10^4} \\ \frac{3^3}{4^2 n^2} &< \frac{1}{10^4} \\ n^2 &> \frac{3^3}{4^2}10^4 \\ n &> \sqrt{3\left(\frac{3 \cdot 10^2}{4}\right)^2} \\ \boxed{n > 75\sqrt{3}}&. \end{aligned}$$

(c) Because $f'' < 0$ on $[1, 4]$, the curve $y = f(x)$ is concave down. The trapezoids all lie below the curve, so T_3 is an $\boxed{\text{underestimate}}$.

3. (30 pts) The following three problems are not related.

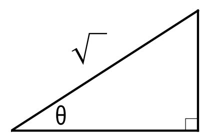
(a) Find the value of $\sin^{-1}\left(\cot\left(\cos^{-1}\left(1/\sqrt{5}\right)\right)\right)$.

(b) Evaluate $\displaystyle\int_0^\infty 6xe^{-2x}\,dx$. Justify any indeterminate limits.

(c) Does $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}\,(1+x^5)}$ converge or diverge? Justify your answer.

Solution:

(a) Let $\theta = \cos^{-1}\left(1/\sqrt{5}\right)$. Then $\cos\theta = 1/\sqrt{5}$. A reference triangle shows that $\cot\theta = 1/2$, so $\sin^{-1}(\cot\theta) = \sin^{-1}(1/2) = \boxed{\pi/6}$.



Note: Because $\cos\theta > 0$, the angle θ is in the first quadrant.

(b) We will use integration by parts with $u = 6x$ and $dv = e^{-2x}\,dx$. Then $du = 6\,dx$ and $v = -\frac{1}{2}e^{-2x}$.

$$\int_0^\infty 6xe^{-2x}\,dx = \lim_{t\to\infty}\int_0^t 6xe^{-2x}\,dx$$

$$= \lim_{t\to\infty}\Big[\underbrace{-3xe^{-2x}\Big|_0^t}_{uv} + \underbrace{\int_0^t 3e^{-2x}\,dx}_{-\int v\,du}\Big]$$

$$= \lim_{t\to\infty} -3xe$$

4. (24 pts) Consider the region R bounded by $y = 4\sqrt{x}$, $x = 0$, and $y = 1$.

(a) Sketch and shade the region R.

(b) Set up but <u>do not evaluate</u> integrals to determine each of the following:

I. The area of R using integration with respect to x.

II. The area of R using integration with respect to y.

III. The volume of the solid when R is rotated about $y = 1$ using the disk method.

Solution:

(a) Note that the curve $y = 4\sqrt{x}$ intersects the line $y = 1$ when $4\sqrt{x} = 1 \implies x = \frac{1}{16}$.

The curve can be represented as $x = \frac{y^2}{16}$, $y \geq 0$.

(b) I. $A = \boxed{\int_0^{1/16} \left(1 - 4\sqrt{x}\right) dx}$

II. $A = \boxed{\int_0^{1} \frac{y^2}{16}\, dy}$

III. $V = \int_a^b \pi r^2\, dx = \boxed{\int_0^{1/16} \pi\left(1 - 4\sqrt{x}\right)^2 dx}$