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 $D = \{x_i, y_i\}$ by coupling to an anti-integrable limit \mathbf{r}_i $\boldsymbol{Y}^{(N)}$ easton, J.D. Meiss $\mathbf{Y}^{(N)}$ *Department of Applied Mathematics, University of Colorado, Boulder, CO 80309-0526, USA*

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Ab t act

Assymplectic twist map near and integrable limit has an invariant set that is conjugate to a full shift on a set of symbols. سام المعالجين المناسلة على السلام العام العالمي العالم العام الأمريكي تعالى العام العام العالمي المعالم التي وال we construct a set of nonzero measure of orbits of the second map that drifts arbitrarily far, even when the coupling is arbitrarily $s_{\rm max}$, these drifting orbits persist near the anti-integrable limit. \sim 2001 Published by Elsevier Science B.V. *MSC:* 37_. 40; 37 40; 37 50

Keywords: S_{cyl} \rightarrow \cdots \cdots \cdots \cdots \cdots \cdots

1. I. **t** \cdot **r ct** \cdot

Are elliptic fixed points of symplectic maps generally stable or unstable or unstable or unstable or unstable o enticing problem. The answer was provided for the planar case by the celebrated work of Kolmogorov, Arnold and Moser (KAM): if the map satisfies a \mathbb{W} in a neighborhood of a fixed point and is not low-order point an resonant, then the point is encircled by a family of invariant curves and hence is stable [2]. For 2n dimensions, \mathbf{A} , theory implies that when a twist condition is satisfied there exists a positive measure family of invariant numbers and n and tori having the fixed point as a limit point as a limit point $n \geq 1$, this family prevent as a limit point of tori does not necessarily prevent as a limit p an orbit starting near the fixed point from wandering far away. The tori should increase α tori should increase α as the distance to the fixed point decreases, and the probability that the probability that the probability th وفيات والرابعة المعدود الذي يتم أندول المعادة وفي محمد التوارد التي ترتيب أن التأم وكان من الرابع والتي تناسب ب z, is by definition unstable if there exists a sequence of points zn → z, and a neighborhood U of z such that the orbit of each point z_n is z_n is possible that such a sequence exists exists even when the orbit is eve of a randomly chosen point near z is likely to remain in U. Arinold gave a famous example of this behavior in 1964 all $\mu_{\rm max}$. Specifically he showed that a nearly integrable hamiltonian flow with more than two degrees of freedom can have orbits that move arbitrarily far from their initial action values. This phenomena has come to be known as \mathbf{A}_i , \mathbf{y}_i as \mathbf{A}_i as \mathbf{y}_i as \mathbf{A}_i as [∗] Corresponding author.

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Fig. 1. Phase space of the map of two uncoupled maps. On the left is a standard map in (x, y) , and on the right is and on integrable twist map in ($\frac{1}{2}$).

diffusion" is to show that such a topological instability "typically" occurs in nearly integrable hamiltonian systems (and perhaps eventually to show that the action drift is indeed diffusive). Arnold's example is special in that it is a periodically time-dependent system with two degrees of freedom, constructed in such a way as to have a normally hyperbolic family of invariant two tori. In this paper, we are concerned with similar phenomena in symplectic maps. A symplectic analogue of Arnold's example is the map on T² × R² defined by x = x + y , ξ = + , y = y − k(1 + h cos ξ)sin x, η = − *kh*(

Fig. 2. $P_{\rm p}$ and the left, α (1) when $h\neq 0$. On the normally hyperbolic cylinder cylinder C suppression C suppression α and show invariant to the invariant circles C_a . The transition C_a or C_a of C_a the heteroclinic point h .

 $\|\cdot\|_{\mathcal{C}^{1,\alpha}}$ that the tori so that a cannot be chosen arbitrarily closely to a. In this case, one in this case, one is case

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3. A_{nti} $t - t$ ab integrable 1

 Δ discrete dynamical system is said to have an anti-integrable limit $3,4,15$, when the dynamics reduces to a full shift on a discrete phase space. For example, the variational principle for the natural system (3) reduces to $=$ * \equiv +91.4887 -291.4887 -thens 220.1(87 -)-17,1.4887 -ces 1.4887 -thens 1.4887 -ces files 87 - A 116 0 0 7.5716 177.3568 5 5256 6640 T 0.5

4. C_u

In this paper, we wish that are coupled to a system near an anti-integrable limit. Denote the system $\mathbb{E}[T]$

that C is not identically zero. Then given any $a < b$, there is a nonzero measure of initial states $(0, 1)$ and a *sequence* $c_t \in C_1 \cup C_2$ (V)– *such that the solution of* (14) *has momenta*, $t = T_2(t_{-1}, t)$ *satisfying* $0 < a$ *and* $\tau > b$ *for some time T.*

P
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$$
P = \{x_1, x_2, \dots, x_n\} \cup \{x_1, x_2, \dots
$$

4.2. Standard example

$$
L(x, x', \ldots') = \frac{1}{2} (x' - x)^2 + \frac{1}{2} (\ldots)^2 + k \ldots x(1 + h \ldots),
$$
\n(15)

 $\left\{\begin{array}{c}\n\mathbf{w} & k > 0 \\
h & \mathbf{w} & h\n\end{array}\right\}$ to \mathbf{A} . $\left\{\begin{array}{ccc}\n\mathbf{A} & \mathbf{w} & \mathbf{w} \\
\mathbf{A} & \mathbf{w} & \mathbf{w}\n\end{array}\right\}$ that $(1), \mathbf{w} & \mathbf{w} & \mathbf{w} & \mathbf{w} & \mathbf{w} & \mathbf{w} & \mathbf{w} \\
\mathbf{w} & \mathbf{w} & \mathbf{w} & \mathbf{w} & \mathbf{w}\n\end{array}$ for h for the set of the set of

or ^t → , then [∗] = 2πm. These points, (0, 2πm) and (π, 2πm), are the fixed points of (16). The only other possibility is that [∗] = (2m + 1)π, in which case the orbit approaches the period two orbit (0, η∗) → (π, η∗). The map (16) is area-preserving since it is generated by a Lagrangian (even though the map is not smooth it is the composition of a pair of shears, each of which is area-preserving). An orbit of a measure-preserving map attracts a set of at most measure zero, because if the stable set had nonzero measure it would eventually be mapped into a ball of arbitrarily small measure about the orbit, violating measure preservation. Thus, the only bounded orbits of (16) are the three orbits that we found and the set of zero measure which limits on these. Therefore, almost all orbits are unbounded. With an effective, area-preserving map of the form (16), an alternative argument based on the Birkhoff ergodic theorem, can also be used. Recall that this theorem states that the time average, g[∗] of a measurable function, g exists almost everywhere, and the space average of g[∗] is equal to the space average of g. The map (16) is periodic both in the coordinate and momentum directions, and so can be considered to be a map on the 2 torus T² = {(ξ , η) : 0 < ξ,η < 2 }. We are interested in average drift rate of the momentum, &η[∗] = (ηt⁺¹ − t) [∗] = lim T →∞ 1 T T −1 t=0 (U (ξt)). Averaging this over all initial conditions, and using the Birkhoff theorem implies that &η∗ = 1 4 ² ^T² U (ξ) ⁼ ² . Whenever the spatial average of the average momentum change per iteration is positive, there must be a nonzero measure of orbits that is unbounded. Thus, whenever the map (16) has nonzero net flux, there is a nonzero measure measure thre drs.012

 \pm U.4. μ and μ and μ and μ and μ are given μ .

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E = \{z \in R : f(z) \notin R\} = R \setminus f^{-1}(R).
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$$
\mu(p(l_a) \cap E_b) = \mu(p(l_a)) - \mu(p(l_a) \cap E_a) \ge \mu(l_a) - \mu(E_a).
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A.
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1 \int_{1}^{1} \frac{1}{e^{x}} \, dx
$$

\n $I_{1} = R \int f_{1}(R),$ $E_{1} = R \int f_{1}^{-1}(R).$
\n(17), $1 \int_{1}^{1} \frac{1}{e^{x}} \, dx$
\n $\frac{1}{2} \int_{1}^{1} \frac{1}{e^{x}}$

L_{nma} **a 4.** *Let* f_t *be a sequence of measure-preserving homeomorphisms, and R a measurable set with incoming sets* I_t *dind exit sets* $t = 1012$ (

5.2. Maps of the cylinder

 \blacksquare

$$
\begin{aligned}\n\frac{\partial}{\partial x} \mathbf{v}_{11} &= \frac{1}{2} \left[\frac{1}{2} \left[\frac{1}{2} \left(\frac{1}{2} \right) \right] + \frac{1}{2} \
$$

 C_a **a** 5. Suppose that f_t *is a sequence of area and end-preserving homeomorphisms of the cylinder, and that the net flux* $t \ge 0$. Let A denote the annulus bounded by the circles $\{y = a\}$ and $\{y = b\}$ where $a < b$. Then, *there is a set of positive measure of orbits that cross the annulus*.

$$
\mathbf{P}_{t+1} \mathbf{P}_{t+1}
$$

 $F_{\rm eff}$, $\frac{1}{2}$, $\frac{1}{$

Fig. 6. Phase space of the standard map for standard map for are map for are many rotational invariant circles.

5.3. Standard map with net flux

 $C_1, C_2, \ldots, C_{n-1}$ generalized standard map (4) on the cylinder $(0, 2) \times R$. The net flux is given by $= V(2) - V(0)$. In general, the force DV can be separated into its mean and oscillatory parts, giving a constant flux and periodic an force, respectively. Thus, the general situation is modeled by the standard map: $x' = x + y'$, $y' = y - k$, $(x) + \frac{1}{2}$. $W_{\text{max}} = 0$, and k $k \ll k_{cr} \approx 0.971635406$ the and ϵ respectively interval in $k \ll k_{cr} \approx 0.971635406$ to a sequence and hence all orbits are $k \ll k_{cr} \approx 0.971635406$ to a sequence and hence all orbits are $k \ll k_{cr} \approx 0.$ bounded. The phase space for k = 0.5 is shown in Fig. 6. By Corollary 5, \mathbf{v}_k is a nonzero is a non measure of orbits that cross from y = 0 to y = 2 . We can visualize these orbits most easily by noting that the s_{1} , s_{2} , s_{3} , s_{4} , s_{5} , s_{6} , s_{7} , s_{8} , s_{10} , s_{11} , s_{12} , s_{13} , s_{14} , s_{15} , s_{16} , s_{17} , s_{18} , s_{19} , s_{10} , s_{11} , s_{12} , s_{13} , s_{14} , s_{15} , s_{16} , s_{1 show that even when is quite small much of the phase space contains unbounded orbits. Periodic orbits of the zero flux standard map are successively destroyed by saddle-center bifurcations as increases. For example, there are fixed points at $y' = 0$ and the two branches of the two branches of

$$
x = \begin{bmatrix} 1 & \cdots & \cdots & 1 \end{bmatrix}
$$

 \mathbf{w} saddle-collide in a saddle-center bifurcation at \mathbf{w} .

6. Pt t c t

In this section, we study the dynamics of the four-dimensional maps given by \mathbf{w} for small . Here, we find it is small . Here, we fi convenient to use the coordinates $z_{t-1} = (x_{t-1}, x_t, t-1, t)$, $\sum_{i=1}^{\infty} \frac{1}{i} \cdot \frac{1}{i} \cdot \frac{1}{i} \cdot \cdots$

Fig. γ_1 and are no rotational map $\kappa = 0.5$, and the $\mathcal{F} = 4^{-2} \times 1000$. The majority orbit (grey region). The majority of majority of majority of majority of majority of majority orbit (grey region).

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x_{t}
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\n
$$
z_{t} = (z_{t-1}) = \frac{-x_{t-1} + 2x_{t} + \frac{1}{t} V(x_{t})A + C(t)}{t}
$$
\n
$$
= t_{t-1} + 2t + W(t) + V(x_{t}) C(t)
$$
\n
$$
y_{t} = 0, y_{t-1} + \frac{1}{t} V(x_{t-1} + \frac{1}{t} V(x
$$

L_{nd}**a 6.** *Suppose that* , *given by* (19), *is a* C^2 *map of* T^4 , *such that* $1 + C(1) \ge 0$. *Then, for any sequence* $\{c_0, c_1, \ldots\}$ *with* $c_t \in \ldots$ (V) \cap **A**, *any initial condition* ($\substack{0, 1}$), *and any* > 0, *there exists an orbit* $Z_t = (x_t, x_{t+1}, t_{t+1}), t \ge 0$ *of such that*

$$
|x_t - c_t| \leq \qquad t \geq 0,
$$

provided

$$
0 \leq \quad <0 = \frac{1}{(4 + a)},
$$

where $(, b) \equiv \quad |c_0| \quad V(c_t \pm \quad).$ (20)

$$
\begin{array}{ll}\n\widetilde{d}(x) \quad V(c_{t+1}^{+}|^{1}) > (4+a) \cdot \widetilde{v}_{1} \dots \overset{a}{\longrightarrow} \cdots \overset{a}{\longrightarrow} \cdots
$$

T 7. *Suppose that satisfies the hypotheses of Lemma* 6. Let $Z_t = (c_t, c_{t+1}, t_{t+1})$ *be an orbit of* 0 *with* $c_t \in \Lambda(V)$. *Then for any* $T \ge 0$ *and* $\lambda > 0$, *there is a* $\lambda > 0$ *such that for all* $\lambda > 0$ *in* (20.5716 0 0 7.5716 439. 1 e_{r+1} is the positive r > 1 is the positive root of r² − wr − 1 = 0, $w = \frac{1}{1-x}(2 + |W(x)|)$. $\mathcal{F}_\mathbf{U}$, $\mathcal{F}_\mathbf{X}$, $\mathcal{F}_\mathbf{X}$, $\mathcal{F}_\mathbf{X}$, $\mathcal{F}_\mathbf{X}$ $|_t| \leq \frac{1}{2} M^2 r^{2t}$. T and ϵ result can be made stronger when $W = 0$, because in this case this case to \mathbb{V}_0 , t^2 , ϵ , ϵ , ϵ , \mathbb{V}_0 , ϵ any \mathcal{I}_1 and any we can construct the so that $|t| \leq \frac{1}{2}$, $t \leq T$. The sound that \Box ${\bf R}$, ${\bf a}$, ${\bf x}$ is not not necessarily hyperbolic and thus one does not expect to be does not expect to be a able to shadow such an orbit for more than a finite interval. Since, the orbit of the anti-integrable system can be chosen to have drifting momenta, this result implies that for small enough , there are orbits of the full system whose momenta grow by an arbitrarily large amount. Note that σ , is true even when the coupling $C(\epsilon)$, is a representation of the coupling small.

6.1. Standard example, continued

Theorem 7 applies to the example (15), and we can compute the bounds explicitly. Since V (x) = k cos x, if we assume δ < π/2, we find = k sin . Since C(ξ) = h cos , we have = 1 − h, so we require h < 1. We can choose the critical points ct ∈ {0, π}, so that the (x, y) orbit does not undergo any rotations. In this case, the acceleration of an arbitrary sequence of critical points is at most a = 2 . Then Lemma 6 applies for [≤] ⁰ ⁼ k(¹ [−] h)sin ⁴ ⁺ ² .

 T_{max} T_{max} $\frac{1}{2}$ $M = k$ _l $\frac{1}{2}$, $M = k$ _l $\frac{1}{2}$, $N = 1$ **b** DB) DB b B Db b b**Jl** ET bDD B

$$
\int_{\mathbb{R}^n} \sup_{\tau \in \mathbb{R}^n} \frac{1}{\|h\|} \frac{1}{\|h\|} \leq \frac{1}{\|h\|} \sup_{\tau \in \mathbb{R}^n} \sup_{\tau \in \mathbb{R}^n} \frac{1}{\|h\|} \sup_{\tau \in \mathbb{R}^n} \sup_{\tau \in \mathbb{R}^n} \sup_{\tau \in \mathbb{R}^n} \frac{1}{\|h\|} \sup_{\tau \in \mathbb{R}^n} \frac{1}{\|h\|}
$$

7. **C**_p **c**₁

We have shown that a twist map that a twist map that is near an anti-integrable limit has many orbits whose ma momenta drift arbitrarily far — even when the coupling is arbitrarily small. Our analysis applies only to the c "a priori chaotic" systems [7] as all the continued orbits from an anti-integrable limit are hyperbolic. The separation of a system into an essentially chaotic degree of freedom that drives an essentially integrable degree of freedom is a common technique used in physical calculations of Arnold diffusion rates $6,9,20$ and α sense, our calculation can be thought of as treating the case of "thick layer" diffusion, rather than the essentially more difficult "thin layer" case, where the chaotic motion is exponentially slow. In the calculations, the coupling — which necessarily affects both degrees of freedom for a symplectic system — is treated by what is called the ما " من كان المعلوم من الرواية التارين من المدينة التاريخ العام المعلوم العام المعلوم التي تعالى التي تعليم المعلوم ال product at the anti-integrable limit. Our results can be viewed as a step along the road to validating the stochastic وراجي الرابع

Of course, we have not provided a solution to the difference \mathbf{A} and \mathbf{A} and \mathbf{A} integrable integrable integrable integrable in near-integrable integrable integrable integrable integrable integrable integra systems? Nor have we show that our drift is actually diffusive. To this end, Moeckel is end, Moeckel and Moecke theorem applies in some cases when the full shift is coupled via a semidirect product to an area-preserving map. The approach we have presented might be generalized might be case when the dynamics in the (x,y) system is system in والتهات فسيرجع فيتعادل النبيلة الفس فالدوم التأريدي الجيئة لأديأ فسيب يتهوم فينهر التهابم من الأبيان التالة وال one is no longer able to drive the drift monotonically, but drift may nevertheless occur. Our approach may also be useful for numerical computations, since the anti-integrable limit is an effective point at which to begin continuation \ldots \ldots 17].

Reference

